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Stability of natural convection in a vertical slot

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The stability of natural convection of a viscous fluid in a vertical slot having isothermal side walls of different temperatures is investigated analytically. Both the conduction and boundary-layer régimes are found to be unstable with respect to stationary disturbances in the form of multicellular secondary flows. Theoretical predictions of the critical Rayleigh number and of the form of the secondary flow are verified by experimental measurements.

1. Introduction

Natural convection in a slot of finite height was first investigated analytically by Batchelor (1954). It was concluded that at low Rayleigh numbers heat is transferred across the slot primarily by conduction. At higher Rayleigh numbers, however, the existence of a new régime consisting of a thin boundary layer around an isothermal core was suggested, with convection the predominant mode of heat transfer. Hereafter these will be referred to as the *conduction régime* and the *boundary-layer régime*, respectively. Interferometric temperature measurements, performed with air by Eckert & Carlson (1961) and with carbon dioxide gas by Mordchelles-Regnier & Kaplan (1963), confirmed the existence of two such flow régimes; however, in the boundary-layer régime a vertical temperature gradient was observed in the core. The same behaviour was found in high Prandtl number fluids by Elder (1965), who measured the velocity field as well as the temperature field. In addition, Elder discovered a multicellular secondary flow which is presumably due to the type of instability considered in this paper.

The stability of natural convection in the conduction régime was first considered by Gershuni (1953), who obtained highly approximate curves of neutral stability for the case of stationary disturbances. Recently this work has been extended by Birikh (1967) and by Rudakov (1967). Ostrach & Maslen (1961) discussed the stability of the same flow with respect to travelling waves of the Tollmien–Schlichting type but did not present a complete solution. However, the instability of this flow with respect to travelling waves of very long wavelength was shown in the thesis of Yuan (1966).

To date, the stability of the boundary-layer régime has not been treated analytically; also, no direct experimental verification of the instability of the conduction régime has been published. The primary purpose of this paper is to report an investigation into these aspects of the problem.

Approximate formulations of both régimes of natural convection in tall, narrow slots are presented. The linearized equations governing small disturbances of these flows are derived and an appropriate extension of Squire's theorem is given. The analysis is confined to consideration of instabilities in the form of cellular secondary flows. Through the use of Galerkin's method curves of neutral stability and secondary-flow stream patterns are determined for each régime.

The theoretical results are qualitatively verified by experimentation for two configurations, one in the conduction régime and one in the boundary-layer régime. This is accomplished by simultaneously measuring the wall temperatures and taking streak photographs of the flow.

2. Base flow

2.1. *Formulation*

Consider the steady two-dimensional natural convection of a viscous fluid in the slot depicted in figure 1. If the Boussinesq approximation is made, the equations

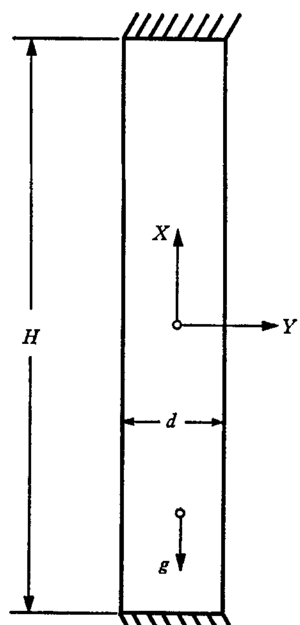

FIGURE 1. Vertical slot.

which govern this motion can be written as

$$\frac{DU}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial X} + g\gamma T' + \nu\nabla^2 U, \tag{1}$$

$$\frac{DV}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial Y} + \nu\nabla^2 V, \tag{2}$$

$$\frac{DT'}{Dt} = \kappa\nabla^2 T', \tag{3}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{4}$$

where the fluid properties ν, γ and κ are assumed to be constant and where p' and T' are the deviations of pressure and temperature from their values at the vertical centre plane of the slot.

For large aspect ratios $h = H/d \gg 1$, an approximate formulation valid far from the ends is obtained by assuming the motion to be parallel, $U = U(Y)$, $V = 0$. In terms of the dimensionless variables $u = U/(\nu/d)$, $p = p'/\rho(\nu/d)^2$, $\Theta = (T-T_m)/\Delta T$, $x = X/H$ and $y = Y/d$, equations (1) to (4) then become

$$\frac{\partial^2 u}{\partial y^2} = -\mathscr{G}\Theta + \frac{\partial p}{\partial x}\Big/h, \tag{5}$$

$$u\frac{\partial \Theta}{\partial x} = \frac{h}{\sigma}\left(\left[\frac{\partial^2\Theta}{\partial x^2}\Big/h^2\right] + \frac{\partial^2\Theta}{\partial y^2}\right), \tag{6}$$

$$\frac{\partial p}{\partial y} = 0, \tag{7}$$

where $\mathscr{G} = g\gamma\Delta T d^3/\nu^2$ is the Grashof number, $\sigma = \nu/\kappa$ the Prandtl number and T_m the temperature at the vertical centre plane.

Since $u = u(y)$, (5) and (7) indicate that $\partial\Theta/\partial y$ is a function of y only; hence if $(1/h\sigma) \ll 1$, equation (6), as pointed out by Elder (1965), implies that the dimensionless temperature gradient, $\partial\Theta/\partial x = \beta h$, is independent of x. The temperature field must then be of the form

$$\Theta = \overline{T}(y) + \beta h x, \tag{8}$$

where β is a constant. Since the walls are isothermal, a solution of the form (8) is strictly valid only at $x = 0$.

On the basis of these considerations, the governing equations reduce to

$$\frac{d^2u}{dy^2} + \mathscr{G}\overline{T} = -\beta x + \frac{dp}{dx}\Big/h = \text{constant} \tag{9}$$

and

$$\beta u = \frac{1}{\sigma}\frac{d^2\overline{T}}{dy^2}, \tag{10}$$

which are subject to the boundary conditions

$$u(\pm\tfrac{1}{2}) = 0, \quad \overline{T}(\pm\tfrac{1}{2}) = \pm\tfrac{1}{2} \tag{11}$$

and to the continuity condition

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} u\,dy = 0. \tag{12}$$

The present analysis provides no means for evaluating β since this requires a two-dimensional study. Reference is made instead to the experimental investigations mentioned in the introduction. These indicate, for both high and low Prandtl number fluids, that the temperature gradient in the vertical centre plane of the slot increases, with increasing Rayleigh number, from a value of zero in the conduction régime to an asymptotic value of about $0{\cdot}5/h$ in the boundary-layer régime. Accordingly, (9)–(12) are next solved using these values for β.

2.2. *Conduction régime* ($\beta = 0$)

The solution of (9)–(12) is simply

$$\overline{T} = y, \tag{13}$$

$$u = \mathscr{G}(\tfrac{1}{4}y - y^3)/6, \tag{14}$$

or

$$U/\overline{U} = \tfrac{1}{6}(\tfrac{1}{4}y - y^3), \tag{15}$$

where $\overline{U} = g\gamma\Delta T d^2/\nu$. This solution was first presented by Batchelor (1954).

2.3. *Boundary-layer régime* ($\beta = 0{\cdot}5/h$)

It is now convenient to combine (9) and (10) in the form

$$\frac{d^4u}{dy^4} + 4m^4u = 0, \quad m^4 \equiv \tfrac{1}{4}\beta\mathscr{R}, \tag{16}$$

with

$$u(\pm\tfrac{1}{2}) = 0, \quad \overline{T}(\pm\tfrac{1}{2}) = \pm\tfrac{1}{2}, \tag{17}$$

where $\mathscr{R} = \mathscr{G}\sigma$ is the Rayleigh number. The corresponding temperature and velocity profiles are readily found to be

$$\overline{T} = -\frac{2m^2K}{\mathscr{G}}\left[\frac{\tan(\frac{1}{2}m)}{\tanh(\frac{1}{2}m)}\sin(my)\cosh(my) + \cos(my)\sinh(my)\right] \tag{18}$$

and

$$u = K\left[-\frac{\tan(\frac{1}{2}m)}{\tanh(\frac{1}{2}m)}\cos(my)\sinh(my) + \sin(my)\cosh(my)\right], \tag{19}$$

where

$$K = -\frac{1}{4}\left\{\frac{m^2}{\mathscr{G}}\left[\frac{\tan(\frac{1}{2}m)}{\tanh(\frac{1}{2}m)}\sin(\tfrac{1}{2}m)\cosh(\tfrac{1}{2}m) + \cos(\tfrac{1}{2}m)\sinh(\tfrac{1}{2}m)\right]\right\}^{-1}. \tag{20}$$

These solutions are essentially those given by Elder (1965) and agree quite well with his measurements at the centre of the slot ($x = 0$). This agreement is in accord with Gill's (1966) remarks.

3. Stability analysis

3.1. *Formulation*

Following the usual approach of the linearized theory of hydrodynamic stability the perturbations

$$\left.\begin{aligned} &u = \overline{u} + \epsilon u', \quad v = \epsilon v', \quad w = \epsilon w', \\ &T = \overline{T} + \epsilon T' \quad \text{and} \quad p = \overline{p} + \epsilon p' \end{aligned}\right\} \tag{21}$$

are introduced into the governing equations. Here bars indicate base flow quantities and ϵ is a small constant parameter. The resulting system accepts solutions of the form

$$q'(x, y, z, t) = q(y)\exp\{i(\alpha x + \beta z - \alpha ct)\}, \tag{22}$$

whose real parts are considered to have physical significance and which may be Fourier components of disturbances of more general structure. The wave-numbers, α and β, are real so that the solution remains bounded as x and y become infinite. The wave speed, c, is in general complex. In terms of (22) the disturbance equations are

$$i\alpha(\bar{u}-c)u+(D\bar{u})v = -i\alpha p+(1/\mathscr{G})\{(D^2-\alpha^2-\beta^2)u+T\}, \tag{23}$$

$$i\alpha(\tilde{u}-c)v = -Dp+(1/\mathscr{G})(D^2-\alpha^2-\beta^2)v, \tag{24}$$

$$i\alpha(\bar{u}-c)w = -i\beta p+(1/\mathscr{G})(D^2-\alpha^2-\beta^2)w, \tag{25}$$

$$i\alpha(\bar{u}-c)T+(D\bar{T})v = (1/\mathscr{R})(D^2-\alpha^2-\beta^2)T, \tag{26}$$

$$u = v = w = T = 0 \quad \text{at} \quad y = \pm\tfrac{1}{2}, \tag{27}$$

where D denotes differentiation with respect to y. It has been assumed that the thermal conductivity of the walls is much greater than that of the fluid. Velocities are referred to $\bar{U} = g\gamma\Delta T\, d^2/\nu$, lengths to the slot width d, temperatures to ΔT, and pressures to $\rho\bar{U}^2$. This scheme of non-dimensionalization brings out the role of the Grashof number as a Reynolds number for the base flow and that of the Rayleigh number as the corresponding Péclet number.

Squire's (1933) theorem can be extended to apply to this problem by noting that the transformations

$$\begin{aligned} \tilde{\alpha}\tilde{u} &= \alpha u+\beta w, & \tilde{p}\tilde{\mathscr{G}} &= p\mathscr{G} \\ \tilde{v} &= v, & \tilde{\alpha}^2 &= \alpha^2+\beta^2, \\ \tilde{\alpha}\tilde{\mathscr{G}} &= \alpha\mathscr{G} & \tilde{\alpha}\tilde{T} &= \alpha T, \\ \tilde{\alpha}\tilde{\mathscr{R}} &= \alpha\mathscr{R}, & \tilde{c} &= c \end{aligned}$$

reduce the system (23)–(27) to that of an equivalent two-dimensional problem for which $\tilde{\mathscr{G}} \leqslant \mathscr{G}$. Hence it is sufficient to consider only two-dimensional disturbances in determining the critical state of neutral stability.

The absence of a preferred direction for wave travel suggests the possibility of the occurrence of a stationary instability, and the multicellular secondary flows and temperature fields observed by Elder (1965) and Mordchelles-Regnier & Kaplan (1963), respectively, provide experimental evidence of such modes. It will later be proved, under certain restrictions, that the wave speed, c, of the disturbance must in fact be zero at neutral stability. The analysis is therefore restricted to this case.

Upon introduction of a stream function

$$\psi' = \phi(y)\exp\{i\alpha(x-ct)\}, \tag{28}$$

such that

$$u = (D\phi)\exp\{i\alpha(x-ct)\} \quad \text{and} \quad v = -i\alpha\phi\exp\{i\alpha(x-ct)\}, \tag{29}$$

the two-dimensional form of (23)–(27) can be reduced, for zero wave speed, to

$$\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi+i\alpha\mathscr{G}(\alpha^2\bar{u}\phi+\bar{u}''\phi-\bar{u}\phi'')+T' = 0, \tag{30}$$

$$T''-\alpha^2T+i\alpha\mathscr{R}(\bar{T}'\phi-\bar{u}T) = 0, \tag{31}$$

$$\phi(\pm\tfrac{1}{2}) = \phi'(\pm\tfrac{1}{2}) = T(\pm\tfrac{1}{2}) = 0. \tag{32}$$

Equation (30) is the Orr–Sommerfeld equation, which is coupled with the thermal disturbance equation (31). Such a system was first given by Gershuni (1953).

3.2. *Solution*

The eigenvalues of the system (30)–(32), which specify the states of neutral stability, form a relationship, $f(\mathscr{G}, \sigma, \alpha) = 0$, which must now be determined. Since the non-self-adjointness of the differential system precludes a classical variational formulation, another approximate solution technique, the Galerkin method, is used, largely because of its simplicity. Accordingly, the disturbance stream function and temperature are expanded as

$$\phi = \sum_{n=1}^{N} a_n \phi_n, \tag{33}$$

$$T = \sum_{n=1}^{N} b_n T_n, \tag{34}$$

where ϕ_n and T_n individually satisfy the boundary conditions (32), and (30) and (31) are orthogonalized with respect to ϕ_n and T_n:

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} L_1\left(\sum_{n=1}^{N} a_n \phi_n, \sum_{n=1}^{N} b_n T_n\right) \phi_n dy = 0, \tag{35}$$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} L_2\left(\sum_{n=1}^{N} a_n \phi_n, \sum_{n=1}^{N} b_n T_n\right) T_n dy = 0, \tag{36}$$

where L_1 and L_2 represent the linear operators of (30) and (31) respectively.

Because the disturbance boundary conditions are symmetric and the base flow under consideration is antisymmetric, the solutions of the disturbance equations are expected to have simple symmetry. Yet inspection of (30) and (31) indicates that since $\bar{u}$ and $\bar{T}$ are odd functions of y the solutions cannot be solely even or solely odd functions of y. However, solutions of the form

$$\phi = \Phi_e + i\Phi_0, \quad T = \tau_0 + i\tau_e, \tag{37}$$

or

$$\phi = \Phi_0 + i\Phi_e, \quad T = \tau_e + i\tau_0, \tag{38}$$

where Φ_e and τ_e denote real even functions of y and Φ_0 and τ_0 denote real odd functions of y, display symmetry and are consistent with the disturbance equations. It is immaterial which of these forms is considered. It can be proved (see appendix) that, for solutions of this form, the real part of the wave speed must be zero.

The function ϕ, which must vanish along with its first derivative at the boundaries, is expanded in terms of the solutions of the characteristic value problem

$$f^{\mathrm{iv}} = a^4 f, \quad f(\pm\tfrac{1}{2}) = f'(\pm\tfrac{1}{2}) = 0. \tag{39}$$

These functions, chosen somewhat arbitrarily and largely for reasons of simplicity, can be written in normalized form as

$$\mathscr{C}_m(y) = \frac{\cosh(\lambda_m y)}{\cosh(\lambda_m/2)} - \frac{\cos(\lambda_m y)}{\cos(\lambda_m/2)} \tag{40}$$

and

$$\mathscr{S}_m(y) = \frac{\sinh(\mu_m y)}{\sinh(\mu_m/2)} - \frac{\sin(\mu_m y)}{\sin(\mu_m/2)} \quad (m = 1, 2, \ldots), \tag{41}$$

where λ_m and μ_m are roots of

$$\tanh(\lambda/2)+\tan(\lambda/2)=0 \quad \text{and} \quad \coth(\mu/2)-\cot(\mu/2)=0. \tag{42}$$

This orthogonal set of functions has been tabulated and discussed by Harris & Reid (1958) and Reid & Harris (1958).

T, which must satisfy only two boundary conditions, is expanded in terms of

$$\cos_m(y)=\cos(\rho_m y) \quad \text{and} \quad \sin_m(y)=\sin(\kappa_m y), \tag{43}$$

where $$\rho_m=(2m-1)\pi \quad \text{and} \quad \kappa_m=2m\pi \quad (m=1,2,\ldots). \tag{44}$$

In accordance with the aforementioned symmetry considerations, the solution of the system (30)–(32) is expanded as

$$\phi(y)=\sum_{n=1}^{N}[a_n\mathscr{C}_n(y)+ib_n\mathscr{S}_n(y)], \tag{45}$$

$$T(y)=\sum_{n=1}^{N}[d_n\sin_n(y)+ie_n\cos_n(y)], \tag{46}$$

where a_n, b_n, d_n and e_n are real constants. Substitution of these series into (30) and (31) and application of the orthogonalization criteria (35) and (36) yield a secular equation which can be represented as

$$\begin{vmatrix} X_{11} & X_{21} & \cdot & X_{N1} \\ X_{12} & X_{22} & \cdot & X_{N2} \\ \cdot & \cdot & \cdot & \cdot \\ X_{1N} & X_{2N} & \cdot & X_{NN} \end{vmatrix}=0. \tag{47}$$

The elemental matrix, X_{mn}, from which the determinant of (47) is constructed, is defined by

$$X_{mn}=\begin{bmatrix} A_{mn} & -\alpha\mathscr{G}B_{mn} & C_{mn} & 0 \\ \alpha\mathscr{G}D_{mn} & E_{mn} & 0 & F_{mn} \\ \alpha\mathscr{R}G_{mn} & 0 & -\alpha\mathscr{R}H_{mn} & I_{mn} \\ 0 & -\alpha\mathscr{R}J_{mn} & K_{mn} & \alpha\mathscr{R}L_{mn} \end{bmatrix}, \tag{48}$$

where

$$\begin{aligned}
A_{mn} &= (C_m^{\text{iv}}\,|\,C_n)-2\alpha^2(C_{mn}''\,|\,C_n)+\alpha^4(C_m\,|\,C_n), \\
B_{mn} &= (S_m\,|\,\alpha^2\bar{u}+\bar{u}''\,|\,C_n)-(S_m''\,|\,\bar{u}\,|\,C_n), \\
C_{mn} &= (\sin_m'\,|\,C_n), \\
D_{mn} &= (C_m\,|\,\alpha^2\bar{u}+\bar{u}''\,|\,S_n)-(C_m''\,|\,\bar{u}\,|\,S_n), \\
E_{mn} &= (S_m^{\text{iv}}\,|\,S_n)-2\alpha^2(S_m''\,|\,S_n)+\alpha^4(S_m\,|\,S_n), \\
F_{mn} &= (\cos_m'\,|\,S_n), \\
G_{mn} &= (C_m\,|\,\bar{T}'\,|\,\cos_n), \\
H_{mn} &= (\sin_m\,|\,\bar{u}\,|\,\cos_n), \\
I_{mn} &= (\cos_m''\,|\,\cos_n)-\alpha^2(\cos_m\,|\,\cos_n), \\
J_{mn} &= (S_m\,|\,\bar{T}'\,|\,\sin_n), \\
K_{mn} &= (\sin_m''\,|\,\sin_n)-\alpha^2(\sin_m\,|\,\sin_n), \\
L_{mn} &= (\cos_m\,|\,\bar{u}\,|\,\cos_n),
\end{aligned} \tag{49}$$

and $(f_m \mid g \mid h_n)$ represents the inner product

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} f_m(y) h_n(y) g(y) \, dy.$$

Details of the solution of the secular equation (47) are presented in the next two sections.

3.3. *Conduction régime*

Inspection of the existing literature reveals that natural convection in a slot of aspect ratio between 10 and 100 is in the conduction régime if the Rayleigh number is less than about 3000. The neutral stability curve for such flows is determined by solving the secular equation (47) with (13) and (15) as the base flow. All inner products in the determinant were integrated exactly. The zeros of the determinant were found by searching the $(\mathscr{G}, \alpha)$-plane using a numerical interval-halving technique. These calculations were carried out at The University of Michigan Computing Centre.

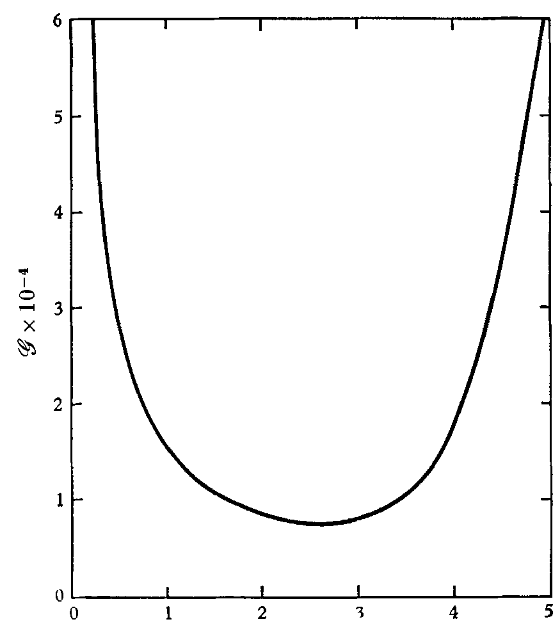

FIGURE 2. Neutral stability curve for the conduction régime.

The convergence of the method was investigated by performing computations with secular determinants of order 4, 8 and 12. The order of the determinant was increased in steps of 4 so that equal numbers of approximating functions for the real and imaginary parts of both the disturbance stream function and temperature were always considered. This avoided artificial weighting of the disturbance momentum or energy equation. The change in the critical Grashof number as the order of the determinant is increased from 8 to 12 is 3·2 %. This procedure was repeated for various Prandtl numbers between 10^{-8} and 10^3, and separately for $\sigma = 0$. The variation of the critical Grashof number with the Prandtl number was found to be less than 0·7 %. Hence the neutral stability curve for stationary disturbances of the conduction régime is presented as a single curve in the $(\mathscr{G}, \alpha)$-plane (figure 2). The critical Grashof number is 7880 at a wave-number of 2·65.

3.4. *Boundary-layer régime*

The lowest value of $\mathscr{R}$ for which the boundary-layer régime exists in a given slot is not in general known; however, inspection of previously cited references reveals that this limiting value is about 8×10^4 for aspect ratios between 10 and

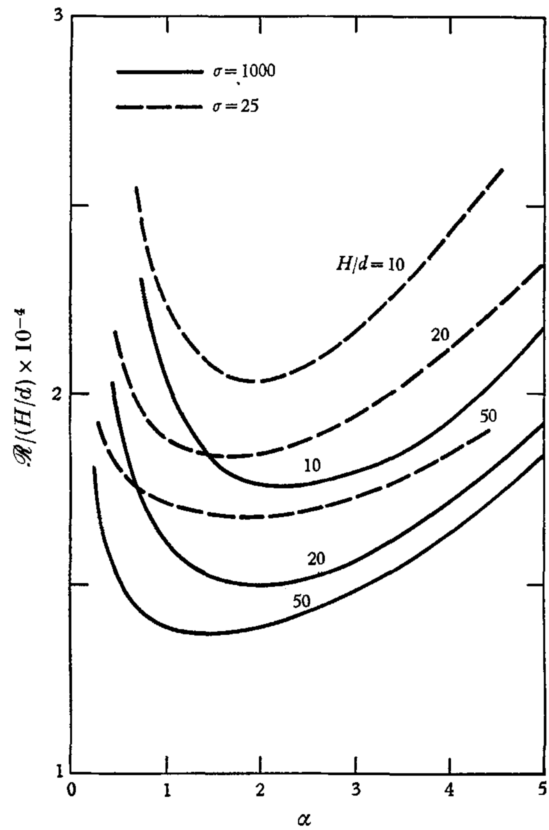

FIGURE 3. Neutral stability curves for the boundary-layer régime.

100. The neutral stability curve for a flow in this régime is determined by solving the secular equation (47) with the appropriately non-dimensionalized form of (18) and (19) as the base flow. The solution of this problem, although formally identical to that of § 3.3, requires a somewhat different numerical procedure. This is because the base flow is now a function of $\mathscr{R}/(H/d)$. All inner products were evaluated by numerical integration and the eigenvalues were determined by searching the $(\mathscr{R}, \alpha)$-plane for zeros of the secular determinant with fixed values of σ and H/d. Convergence studies were carried out using determinants of order 4, 8, 12 and 16 for various values of σ and H/d. The convergence in these cases is not as rapid as in the conduction régime; however, the maximum change in

critical Rayleigh number as the order of the determinant is increased from 12 to 16 is only about 3 %.

In figure 3 neutral stability curves are presented for various values of σ and H/d. In the range of aspect ratios considered here, attempts to obtain solutions with moderate sized determinants were not successful for Prandtl numbers between 1 and 25.

4. Secondary flow

The real, and physically significant, part of the complex stream function (28) is represented by

$$\psi' = \sum_{n=1}^{N} [a_n \mathscr{C}_n(y) \cos(\alpha x) - b_n \mathscr{S}_n(y) \sin(\alpha x)]. \tag{50}$$

The coefficients in (50) are determined, with respect to an arbitrary a_1, through the use of (47). The disturbance stream pattern evaluated in the neighbourhood

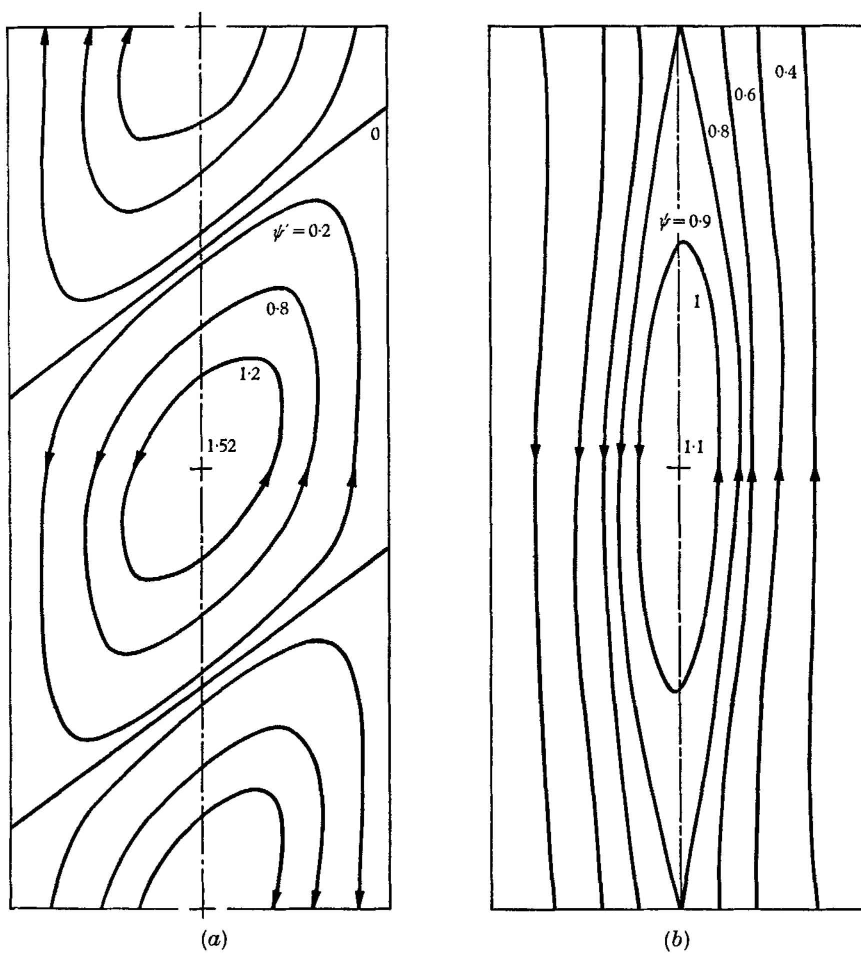

FIGURE 4. Streamline patterns for the conduction régime. $\mathscr{G}$ = 7877: (*a*) disturbance; (*b*) total flow, ϵ = 0·1.

of the critical state of the conduction régime is displayed in figure 4 (*a*). An analogous pattern for the boundary-layer régime ($\mathscr{R} = 3{\cdot}12 \times 10^5, H/d = 20, \sigma = 1000$) is displayed in figure 5 (*a*).

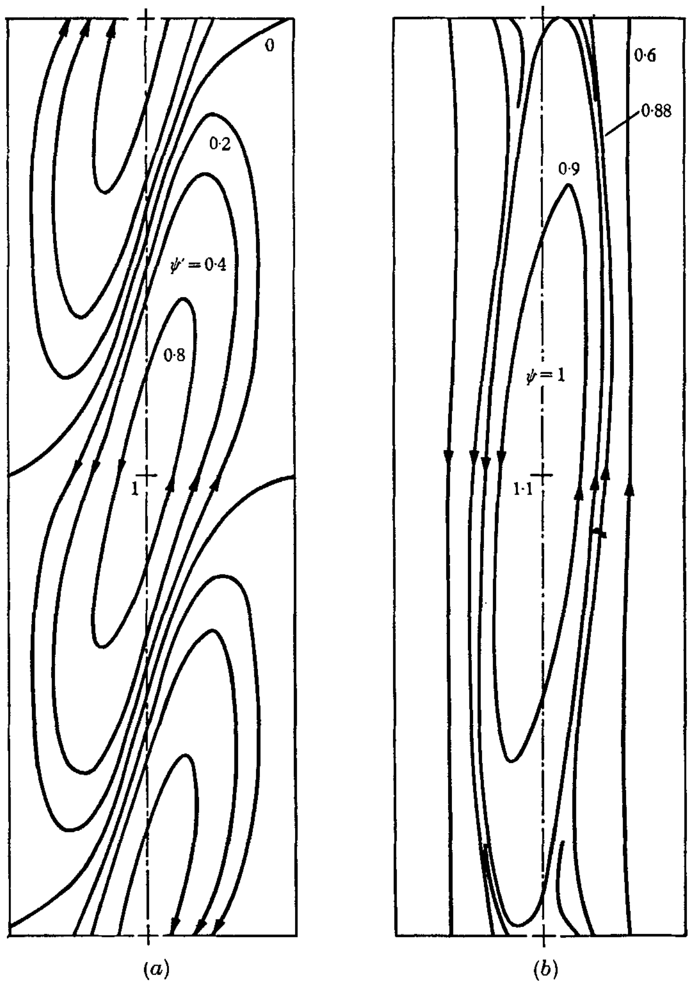

FIGURE 5. Streamline patterns for the boundary-layer régime. $\mathscr{R} = 3{\cdot}12 \times 10^5$, $H/d = 20$, $\sigma = 1000$: (*a*) disturbance; (*b*) total flow, $\epsilon = 0{\cdot}1$.

To provide for comparison with experimental flow visualizations, the disturbance and base stream functions are normalized to maximum values of unity and superimposed as

$$\psi = \overline{\psi} + \epsilon\psi', \tag{51}$$

with ϵ arbitrarily assigned the value 0·1. The resulting stream patterns are displayed in figures 4 (*b*) and 5 (*b*).

5. Experiment

A slot formed by a rectangular plexiglass frame bounded by aluminium side walls was constructed. Each side wall also formed one side of a baffled chamber through which water, used to heat or cool the wall, was circulated. The dimensions of the slot were varied by changing the plexiglass frame. Twelve sheathed copper-constantan thermocouples were imbedded in the aluminium plates. Their outputs were monitored by a printing recorder and measured with a precision potentiometer. Further details are given by Vest (1967).

The conduction régime was established in air ($\sigma \simeq 0{\cdot}71$) contained in a slot of dimensions $H = 25$ in., $d = 0{\cdot}75$ in. and $B = 4{\cdot}5$ in., where B is the depth in the direction normal to the flow. To visualize the motion, a small amount of cigarette smoke was introduced at the top of the slot. The smoke particles were entrained by the fluid and were made visible by illuminating a vertical plane, approximately $\frac{1}{2}$ in. deep, in the centre of the slot. The wall temperature difference was very slowly increased and the onset of cellular motion was observed.

The critical Grashof number, assumed to correspond to the first cell formation to be observed far from the ends of the slot, was found to be $8700 \pm 10\,\%$ with a corresponding wave-number of about $2{\cdot}74$. The error margin indicated for the critical Grashof number is somewhat subjective since it is largely limited by the visualization process rather than the quantitative measurements. Figure 6(*a*), plate 1, is a photograph of the cellular pattern at a Grashof number which is $9\,\%$ in excess of the critical value. These cells may be compared with the one in figure 4(*b*).

The boundary-layer régime was established in a silicone oil, DC 200/100 ($\sigma \simeq 900$), contained in a slot of dimensions $H = 25$ in., $d = 1{\cdot}25$ in. and $B = 4{\cdot}5$ in. This experiment corresponds to the analysis for $H/d = 20$, $\sigma = 1000$. The motion was visualized by suspending small aluminium particles in the fluid and illuminating a vertical plane. The critical Rayleigh number was found to be $3{\cdot}7 \times 10^5 \pm 10\,\%$ with a corresponding wave-number of about $3{\cdot}5$. It may be noted that while the critical Rayleigh number is in reasonable agreement with the theoretical prediction of $3{\cdot}12 \times 10^5$, the wave-number is considerably different from the predicted value of $1{\cdot}85$.

Figure 6(*b*), plate 1, is a photograph of a cell at a Rayleigh number which is $20\,\%$ in excess of the critical value. This cell may be compared with that of figure 5(*b*).

6. Discussion

It is concluded that the natural-convective flow in a vertical slot is unstable with respect to certain stationary modes of disturbance, which have been analytically investigated and experimentally observed. This phenomenon is rather unique in that the stationary instability, which is made possible by the antisymmetry of the flow, occurs in the plane of a base flow, unlike the classical cellular instabilities such as those of Bénard, Taylor and Görtler. A consequence of the presence of the base flow is the tilting of the cells with respect to the walls of the slot.

After the analysis of the conduction régime had been completed, a closely related work by Birikh (1967) was published in English translation. In it is reported an analytical study of the eigenvalue spectrum for disturbances in the conduction régime of a fluid with zero Prandtl number. The lowest eigenvalue agrees with the critical Grashof number of the present study to within 0·7 %, even though a different set of approximating functions is utilized in a Galerkin analysis. More recently this study has been extended by Rudakov (1967), who considered the conduction régime for the Prandtl number range $0{\cdot}01 \leqslant \sigma \leqslant 10$. The critical Grashof number was found to be a weak function of σ, deviating from our value by ± 7 % in the centre of this range, but asymptotically approaching it at either end. The weakness of the σ dependence suggests that the buoyancy forces play only a minimal role in the destabilization of the flow. This interpretation was substantiated when the limiting case of zero Prandtl number was considered. In this case, the system of disturbance equations reduces to the classical Orr–Sommerfeld equation with no buoyancy force term. The corresponding critical Grashof number is 7932, which differs from the value reported above by only 0·7 %. In the boundary-layer régime, however, the Prandtl number plays a dual role, appearing in the parameter m which determines the base flow, as well as in the Rayleigh number which appears explicitly in the disturbance energy equation. Consideration of the results shown in figure 3 reveals that the critical Grashof number for this régime exhibits a strong dependence on σ in addition to that on m. This unanticipated result implies a fundamental difference between the destabilization mechanism in the conduction régime and that in the boundary-layer régime. The former appears to be purely hydrodynamic while the latter appears to involve the thermal buoyancy force. This interpretation was further supported by introducing the boundary-layer régime base velocity profile into the Orr–Sommerfeld equation alone, with the buoyancy force term excluded. No neutrally stable states were found in the vicinity of those presented in figure 3, although the conduction régime results were reproduced when the Prandtl number was small ($m \rightarrow 0$).

In general, secular determinants of only moderate order were required for convergence to the critical eigenvalues within a few percent. This is not the case, however, for the boundary-layer régime in slots of aspect ratio less than 50, for which solutions were not obtainable with Prandtl numbers between 1 and 25. At this stage, this difficulty appears to be of numerical origin, although it may be noted that in this range the transition from the conduction to the boundary-layer régime occurs and the assumed base velocity and temperature profiles must be in error. At lower values of σ the solution for the boundary-layer régime approaches that for the conduction régime, as expected.

The experimental results for the conduction régime were in better agreement with the analysis than were those for the boundary-layer régime. This may well be due to the more approximate nature of the base solution for the boundary-layer régime.

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Appendix

It will now be proved that the real part of the wave speed, c_r, must be zero if the solutions, ϕ and T, are of the form (37) or (38).

The equations and boundary conditions governing a small disturbance with complex wave speed c are

$$\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi + i\alpha\mathscr{G}(\bar{u}''\phi - (\bar{u}-c)(\phi''-\alpha^2\phi)) + T' = 0, \tag{A 1}$$

$$T'' - \alpha^2 T + i\alpha\mathscr{R}(\bar{T}'\phi - (\bar{u}-c)T) = 0, \tag{A 2}$$

$$\phi(\pm\tfrac{1}{2}) = \phi'(\pm\tfrac{1}{2}) = T(\pm\tfrac{1}{2}) = 0, \tag{A 3}$$

where

$$\bar{u}(y) = -\bar{u}(-y), \quad \bar{T}(y) = -\bar{T}(-y). \tag{A 4}$$

Equations (A 1) and (A 2) are multiplied respectively by ϕ^* and T^*, the complex conjugates of ϕ and T, and integrated over $[-\frac{1}{2}, +\frac{1}{2}]$. Suitable integrations by parts and utilization of conditions (A 3) and (A 4) yield

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}}(|\phi''|^2 + 2\alpha^2|\phi'|^2 + \alpha^4|\phi|^2)\,dy + i\alpha\mathscr{G}\left\{\int_{-\frac{1}{2}}^{+\frac{1}{2}}((\bar{u}''+\alpha^2\bar{u})|\phi|^2 - \bar{u}\phi''\phi^*)\,dy - c\int_{-\frac{1}{2}}^{+\frac{1}{2}}(|\phi'|^2+\alpha^2|\phi|^2)\,dy\right\} + \int_{-\frac{1}{2}}^{+\frac{1}{2}} T'\phi^*\,dy = 0 \tag{A 5}$$

and

$$-\int_{-\frac{1}{2}}^{+\frac{1}{2}}(|T'|^2 + \alpha^2|T|^2)\,dy + i\alpha\mathscr{R}\left\{\int_{-\frac{1}{2}}^{+\frac{1}{2}}(\bar{T}'\phi T^* - \bar{u}|T|^2)\,dy + c\int_{-\frac{1}{2}}^{+\frac{1}{2}}|T|^2 dy\right\} = 0. \tag{A 6}$$

For the particular case in which ϕ and T are of the form (37) the following integrals are obtained:

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}}((\bar{u}''+\alpha^2\bar{u})|\phi|^2 - \bar{u}\phi''\phi^*)\,dy = i\int_{-\frac{1}{2}}^{+\frac{1}{2}}\bar{u}(\Phi_0''\Phi_e - \Phi_e''\Phi_0)\,dy, \tag{A 7}$$

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} T'\phi^*\,dy = \int_{-\frac{1}{2}}^{+}(\tau_0'\Phi_e + \tau_e'\Phi_0)\,dy \tag{A 8}$$

and

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}}(\bar{T}'\phi T^* - \bar{u}|T|^2)\,dy = i\int_{-\frac{1}{2}}^{+\frac{1}{2}}\bar{T}'(\Phi_0\tau_0 - \Phi_e\tau_e)\,dy. \tag{A 9}$$

Using (A 7)–(A 9), equations (A 5) and (A 6) can be rewritten in terms of their explicit real and imaginary parts. The imaginary parts of (A 5) and (A 6) are respectively

$$-\alpha \mathscr{G} c_r \int_{-\frac{1}{2}}^{+\frac{1}{2}} (|\phi'|^2 + \alpha^2 |\phi|^2)\, dy = 0 \tag{A 10}$$

and

$$\alpha \mathscr{R} c_r \int_{-\frac{1}{2}}^{+\frac{1}{2}} |T|^2\, dy = 0. \tag{A 11}$$

Since both of these integrals are positive definite, it is necessary that

$$c_r = 0. \tag{A 12}$$

The same conclusion is obtained if solutions of the form (38) are considered.

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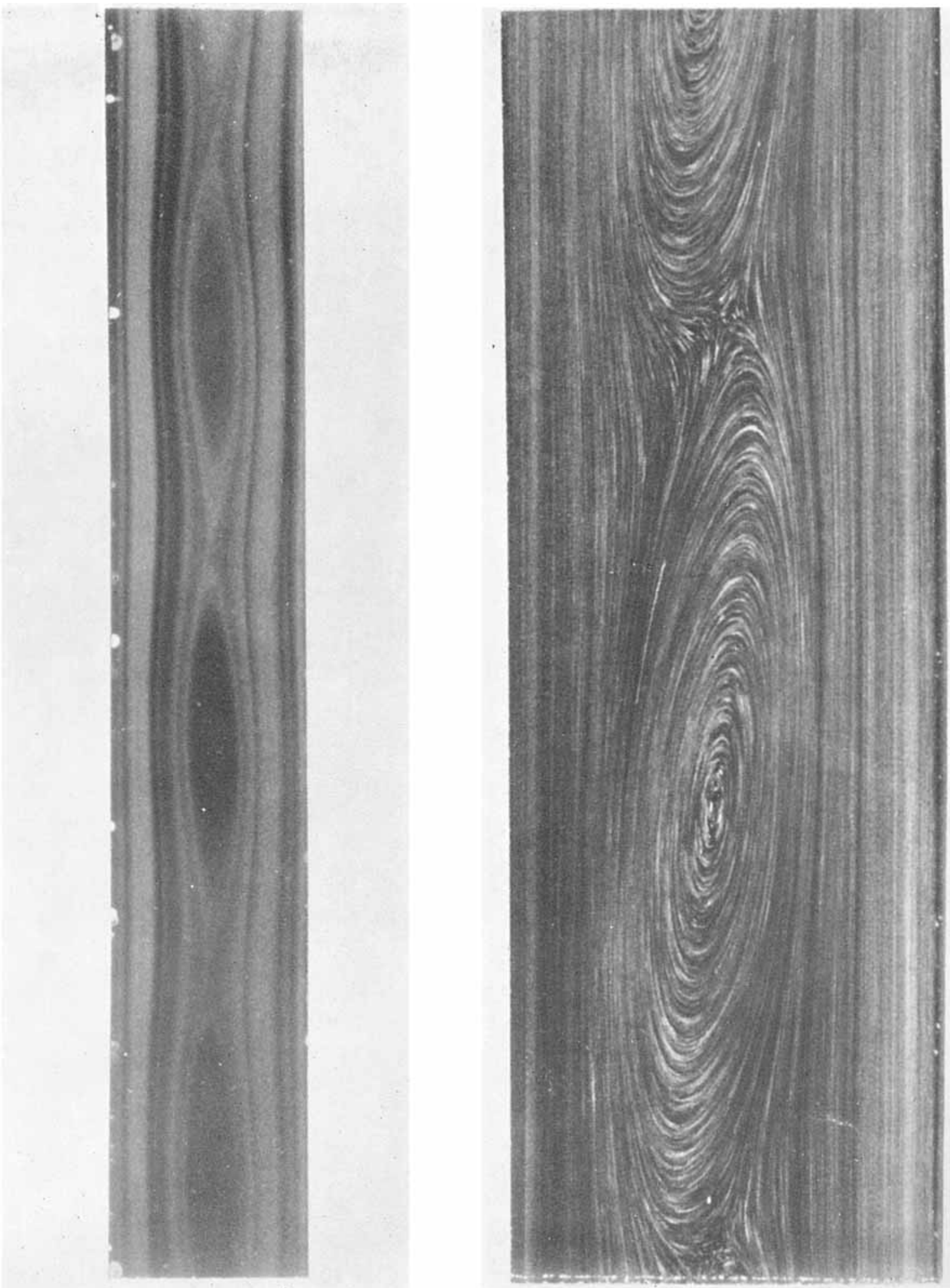

(*a*) (*b*)

FIGURE 6. Streak photographs of the secondary flow: (*a*) conduction régime, $\mathscr{G} = 9500$, $H/d = 33$, $\sigma = 0{\cdot}71$; (*b*) boundary-layer régime, $\mathscr{R} \cong 4{\cdot}5 \times 10^5$, $H/d = 20$, $\sigma \cong 900$.